

ULB/229/CQ/97/5

# $C_\lambda$ -extended harmonic oscillator and (para)supersymmetric quantum mechanics

C. Quesne <sup>\*</sup>, N. Vansteenkiste <sup>†</sup>

*Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles,  
Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium*

## Abstract

$C_\lambda$ -extended oscillator algebras are realized as generalized deformed oscillator algebras. For  $\lambda = 3$ , the spectrum of the corresponding bosonic oscillator Hamiltonian is shown to strongly depend on the algebra parameters. A connection with cyclic shape invariant potentials is noted. A bosonization of PSSQM of order two is obtained.

PACS: 03.65.Fd

Keywords: deformed oscillator algebras, supersymmetric quantum mechanics, parastatistics

To be published in Phys. Lett. A

---

<sup>\*</sup>Directeur de recherches FNRS. E-mail: cquesne@ulb.ac.be

<sup>†</sup>Boursier ULB. E-mail: nvsteen@ulb.ac.be

# 1 Introduction

During the last few years, exotic quantum statistics have received considerable attention in the literature. In two spatial dimensions, one can have anyonic statistics [1], interpolating between bosonic and fermionic ones, which has been proposed as a mechanism for the fractional quantum Hall effect and high- $T_c$  superconductivity [2]. In higher dimensions, parabosonic and parafermionic statistics have been suggested as generalizations of the standard bosonic and fermionic ones, describing representations of the permutation group that are neither completely symmetrical nor completely antisymmetrical [3].

On the other hand, supersymmetric quantum mechanics (SSQM) has established a nice symmetry between bosons and fermions [4]. Furthermore, when supplemented with the concept of shape invariance [5], it has also provided a powerful method of generating exactly solvable quantum mechanical models. Devising new approaches to construct shape invariant potentials is still under current investigation (for a recent review see Ref. [6]). Among the most recent advances in this field, one may quote the introduction of cyclic shape invariant potentials by Sukhatme *et al.* [7], generalizing a previous work of Gangopadhyaya and Sukhatme [8].

In view of the SSQM success, there have been various attempts to extend its formalism to some of the exotic statistics. Combining for instance bosons with parafermions (instead of fermions) has led to parasupersymmetric quantum mechanics (PSSQM), existing in the literature in two (generally) inequivalent forms, due to Rubakov-Spiridonov [9], and Beckers-Debergh [10], respectively.

The development of quantum groups and quantum algebras [11] during the last decade has proved very useful in connection with such problems. Various deformations and extensions of the oscillator algebra (for a recent review see Ref. [12]) have indeed been applied to the description of systems with nonstandard statistics [13, 14, 15, 16, 17], the algebraic formulation of some quantum integrable models [18, 19, 20], and the bosonization of SSQM [17, 20, 21].

The purpose of the present letter is to introduce new  $C_\lambda$ -extended oscillator algebras, where  $C_\lambda$  denotes a cyclic group of order  $\lambda$ , and to show their usefulness in both SSQM and PSSQM frameworks.

## 2 $C_\lambda$ -extended oscillator algebras

A  $C_\lambda$ -extended oscillator algebra  $\mathcal{A}^{(\lambda)}$ , where  $\lambda$  may take any value in the set  $\{2, 3, 4, \dots\}$ , is defined as an algebra generated by the operators  $I$ ,  $a^\dagger$ ,  $a = (a^\dagger)^\dagger$ ,  $N = N^\dagger$ , and  $T = (T^\dagger)^{-1}$ , satisfying the relations

$$\begin{aligned} [N, a^\dagger] &= a^\dagger, & [N, T] &= 0, & T^\lambda &= I, \\ [a, a^\dagger] &= I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, & a^\dagger T &= e^{-2\pi i/\lambda} T a^\dagger, \end{aligned} \quad (1)$$

together with their Hermitian conjugates. Here  $\kappa_\mu$ ,  $\mu = 1, 2, \dots, \lambda - 1$ , are some complex parameters restricted by the conditions  $\kappa_\mu^* = \kappa_{\lambda-\mu}$  (so that there remain altogether  $\lambda - 1$  independent real parameters), and  $T$  is the generator of a cyclic group of order  $\lambda$ ,  $C_\lambda = \{I, T, T^2, \dots, T^{\lambda-1}\}$  (or, more precisely, the generator of a unitary representation thereof). As usual,  $N$ ,  $a^\dagger$ , and  $a$  will be called number, creation, and annihilation operators, respectively.

As well known [22],  $C_\lambda$  has  $\lambda$  inequivalent unitary irreducible matrix representations  $\Gamma^\mu$ ,  $\mu = 0, 1, \dots, \lambda - 1$ , which are one-dimensional, and such that  $\Gamma^\mu(T^\nu) = \exp(2\pi i \mu \nu / \lambda)$  for any  $\nu = 0, 1, \dots, \lambda - 1$ . The projection operator on the carrier space of  $\Gamma^\mu$  may be written as

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} (\Gamma^\mu(T^\nu))^* T^\nu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-2\pi i \mu \nu / \lambda} T^\nu, \quad (2)$$

and conversely  $T^\nu$ ,  $\nu = 0, 1, \dots, \lambda - 1$ , may be expressed in terms of the  $P_\mu$ 's as

$$T^\nu = \sum_{\mu=0}^{\lambda-1} e^{2\pi i \mu \nu / \lambda} P_\mu. \quad (3)$$

The algebra defining relations (1) may therefore be rewritten in terms of  $I$ ,  $a^\dagger$ ,  $a$ ,  $N$ , and  $P_\mu = P_\mu^\dagger$ ,  $\mu = 0, 1, \dots, \lambda - 1$ , as

$$\begin{aligned} [N, a^\dagger] &= a^\dagger, & [N, P_\mu] &= 0, & \sum_{\mu=0}^{\lambda-1} P_\mu &= I, \\ [a, a^\dagger] &= I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, & a^\dagger P_\mu &= P_{\mu+1} a^\dagger, \end{aligned} \quad (4)$$

where we use the conventions  $P_\lambda \equiv P_0$ ,  $P_{-1} \equiv P_{\lambda-1}$  (and similarly for other operators or parameters indexed by  $\mu$ ). By definition of projection operators, the  $P_\mu$ 's satisfy the relations  $P_\mu P_\nu = \delta_{\mu,\nu} P_\mu$ . Equation (4) depends upon  $\lambda$  real parameters  $\alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(2\pi i \mu \nu / \lambda) \kappa_\nu$ ,  $\mu = 0, 1, \dots, \lambda-1$ , restricted by the condition  $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$ . Hence, we may eliminate one of them, for instance  $\alpha_{\lambda-1}$ , and denote the algebra by  $\mathcal{A}_{\alpha_0 \alpha_1 \dots \alpha_{\lambda-2}}^{(\lambda)}$ . It will however often prove convenient to work instead with the  $\lambda$  dependent parameters  $\alpha_0, \alpha_1, \dots, \alpha_{\lambda-1}$ .

We may realize  $C_\lambda$  in various ways. Two of the simplest ones use either functions of  $N$ , or functions of spin- $s$  matrices, where  $s \equiv (\lambda-1)/2$ . Here, we will consider the former choice. Hence, in the remainder of this letter, we will assume that  $T$  and (as a consequence of Eq. (2))  $P_\mu$  are given by

$$T = e^{2\pi i N / \lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{2\pi i \nu (N-\mu) / \lambda}, \quad \mu = 0, 1, \dots, \lambda-1, \quad (5)$$

respectively.

With such a choice,  $\mathcal{A}_{\alpha_0 \alpha_1 \dots \alpha_{\lambda-2}}^{(\lambda)}$  may be considered as a generalized deformed oscillator algebra (GDOA)  $\mathcal{A}(G(N))$ , with  $G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu$ , and  $P_\mu$  given by Eq. (5) [23]. For any GDOA, one may define a so-called structure function  $F(N)$ , which is the solution of the difference equation  $F(N+1) - F(N) = G(N)$ , such that  $F(0) = 0$  (see Ref. [12] and references quoted therein). In the present case, a straightforward calculation leads to

$$F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu, \quad \beta_0 \equiv 0, \quad \beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu \quad (\mu = 1, 2, \dots, \lambda-1). \quad (6)$$

It has been shown [24] that GDOAs may have in general various types of unitary irreducible representations according the nature of the  $N$  spectrum, but here we shall only be interested in the bosonic Fock-space representation, wherein

$$a^\dagger a = F(N), \quad a a^\dagger = F(N+1). \quad (7)$$

Its carrier space is spanned by the eigenvectors  $|n\rangle$  of the number operator  $N$ , corresponding to the eigenvalues  $n = 0, 1, 2, \dots$ , where  $|0\rangle$  is assumed to be a vacuum state, i.e.,  $a|0\rangle = 0$ . The eigenvectors can be written as

$$|n\rangle = \mathcal{N}_n^{-1/2} (a^\dagger)^n |0\rangle, \quad n = 0, 1, 2, \dots, \quad (8)$$

where  $\mathcal{N}_n = \prod_{i=1}^n F(i)$ . By writing  $n$  as  $n = k\lambda + \mu$ , where  $\mu \in \{0, 1, \dots, \lambda - 1\}$ , and  $k$  is some nonnegative integer,  $\mathcal{N}_n$  can be expressed in terms of gamma functions as

$$\mathcal{N}_{k\lambda+\mu} = \lambda^{k\lambda+\mu} \left( \prod_{\nu=0}^{\mu} \Gamma(k+1+\bar{\beta}_{\nu}) \right) \left( \prod_{\nu'=\mu+1}^{\lambda-1} \Gamma(k+\bar{\beta}_{\nu'}) \right) \left( \prod_{\nu''=1}^{\lambda-1} \Gamma(\bar{\beta}_{\nu''}) \right)^{-1}, \quad (9)$$

where  $\bar{\beta}_{\nu} \equiv (\beta_{\nu} + \nu) / \lambda$ . The creation and annihilation operators act upon  $|n\rangle$  in the usual way, i.e.,

$$a^{\dagger}|n\rangle = \sqrt{F(n+1)}|n+1\rangle, \quad a|n\rangle = \sqrt{F(n)}|n-1\rangle, \quad (10)$$

while  $P_{\mu}$  projects on the  $\mu$ th component  $\mathcal{F}_{\mu} \equiv \{|k\lambda + \mu\rangle \mid k = 0, 1, 2, \dots\}$  of the  $\mathbb{Z}_{\lambda}$ -graded Fock space  $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \oplus \mathcal{F}_{\mu}$ . It is obvious that such a Fock-space representation exists if and only if  $F(\mu) > 0$  for  $\mu = 1, 2, \dots, \lambda - 1$ . These conditions imply the following restrictions on the parameters  $\alpha_{\mu}$ ,

$$\sum_{\nu=0}^{\mu-1} \alpha_{\nu} > -\mu, \quad \mu = 1, 2, \dots, \lambda - 1. \quad (11)$$

For the lowest allowed  $\lambda$  value, i.e.,  $\lambda = 2$ , the operator  $T$ , defined in Eq. (5), reduces to the Klein operator  $K = \exp(i\pi N)$ , which in the Fock-space representation becomes  $K = (-1)^N$ . The corresponding operators  $P_0 = \frac{1}{2}(I + (-1)^N)$ , and  $P_1 = \frac{1}{2}(I - (-1)^N)$  project upon the even subspace  $\mathcal{F}_0 = \{|2k\rangle \mid k = 0, 1, 2, \dots\}$ , and the odd subspace  $\mathcal{F}_1 = \{|2k+1\rangle \mid k = 0, 1, 2, \dots\}$  of  $\mathcal{F}$ , respectively. The  $C_2$ -extended oscillator algebra  $\mathcal{A}_{\alpha_0}^{(2)}$  is therefore nothing else than the Calogero-Vasiliev oscillator algebra [18], providing an algebraic formulation of the two-particle Calogero problem [19, 20], an alternative description of parabosons [14], and a bosonization of SSQM [17, 20]. It depends upon a single independent real parameter  $\kappa_1 = \kappa_1^* = \alpha_0 = -\alpha_1$ , restricted by the condition  $\alpha_0 > -1$  in the Fock-space representation.

The next allowed  $\lambda$  value, i.e.,  $\lambda = 3$ , gives rise to a new algebraic structure, the  $C_3$ -extended oscillator algebra  $\mathcal{A}_{\alpha_0\alpha_1}^{(3)}$ , corresponding to  $T = \exp(2\pi i N/3)$ , and

$$\begin{aligned} P_0 &= \frac{1}{3} \left( I + 2 \cos \frac{2\pi}{3} N \right), & P_1 &= \frac{1}{3} \left( I - \cos \frac{2\pi}{3} N + \sqrt{3} \sin \frac{2\pi}{3} N \right), \\ P_2 &= \frac{1}{3} \left( I - \cos \frac{2\pi}{3} N - \sqrt{3} \sin \frac{2\pi}{3} N \right), \end{aligned} \quad (12)$$

projecting on  $\mathcal{F}_0 = \{|3k\rangle \mid k = 0, 1, 2, \dots\}$ ,  $\mathcal{F}_1 = \{|3k+1\rangle \mid k = 0, 1, 2, \dots\}$ , and  $\mathcal{F}_2 = \{|3k+2\rangle \mid k = 0, 1, 2, \dots\}$ , respectively. It depends upon two independent real parameters, which may be taken as the real and imaginary parts of  $\kappa_1$  (with  $\kappa_2 = \kappa_1^*$ ), or as  $\alpha_0 = 2 \Re \kappa_1$ , and  $\alpha_1 = -\Re \kappa_1 - \sqrt{3} \Im \kappa_1$  (with  $\alpha_2 = -\alpha_0 - \alpha_1 = -\Re \kappa_1 + \sqrt{3} \Im \kappa_1$ ), restricted by the conditions  $\alpha_0 > -1$ , and  $\alpha_0 + \alpha_1 > -2$  in the Fock-space representation. For such an algebra, we may write

$$\begin{aligned} [a, a^\dagger] &= I + 2(\Re \kappa_1) \cos \frac{2\pi}{3} N - 2(\Im \kappa_1) \sin \frac{2\pi}{3} N \\ &= I + \alpha_0 P_0 + \alpha_1 P_1 - (\alpha_0 + \alpha_1) P_2. \end{aligned} \quad (13)$$

Similar explicit relations can easily be written down for  $\lambda = 4, 5, \dots$

### 3 $C_\lambda$ -extended oscillator Hamiltonian and supersymmetric quantum mechanics

As usual, we define the bosonic oscillator Hamiltonian associated to the algebra  $\mathcal{A}_{\alpha_0 \alpha_1 \dots \alpha_{\lambda-2}}^{(\lambda)}$ , in appropriate units, as

$$H_0 \equiv \frac{1}{2} \{a, a^\dagger\}. \quad (14)$$

By using Eqs. (4), (6), and (7),  $H_0$  can be rewritten in the equivalent forms

$$H_0 = a^\dagger a + \frac{1}{2} \left( I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \right) = N + \frac{1}{2} I + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu, \quad (15)$$

where the parameters  $\gamma_\mu$  are defined by

$$\gamma_\mu \equiv \frac{1}{2}(\beta_\mu + \beta_{\mu+1}) = \begin{cases} \frac{1}{2}\alpha_0 & \text{if } \mu = 0, \\ \sum_{\nu=0}^{\mu-1} \alpha_\nu + \frac{1}{2}\alpha_\mu & \text{if } \mu = 1, 2, \dots, \lambda-1, \end{cases} \quad (16)$$

and satisfy the relation  $\sum_{\mu=0}^{\lambda-1} (-1)^\mu \gamma_\mu = 0$ .

The eigenvectors of  $H_0$  are the states  $|n\rangle$ , defined in Eqs. (8) and (9), and their eigenvalues are given by

$$E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}, \quad k = 0, 1, 2, \dots, \quad \mu = 0, 1, \dots, \lambda-1. \quad (17)$$

In each  $\mathcal{F}_\mu$  subspace of the  $\mathbb{Z}_\lambda$ -graded Fock space  $\mathcal{F}$ , the spectrum of  $H_0$  is therefore harmonic, but the  $\lambda$  infinite sets of equally spaced energy levels, corresponding to  $\mu = 0$ ,

$1, \dots, \lambda - 1$ , may be shifted with respect to each other by some amounts depending upon the algebra parameters  $\alpha_0, \alpha_1, \dots, \alpha_{\lambda-2}$ , through their linear combinations  $\gamma_\mu$ ,  $\mu = 0, 1, \dots, \lambda - 1$ .

For the Calogero-Vasiliev oscillator, the two infinite sets of energy levels corresponding to  $\mathcal{F}_0$  and  $\mathcal{F}_1$ , respectively, are always shifted with respect to one another by one energy unit, since the relation  $\gamma_0 = \gamma_1$  is valid for any  $\alpha_0$  value. The resulting spectrum is therefore very simple, and coincides with that of a shifted harmonic oscillator.

For  $\lambda \geq 3$ , the situation is entirely different. According to the parameter values, the spectrum may be nondegenerate, or may exhibit some  $(\nu + 1)$ -fold degeneracies above some energy eigenvalue, where  $\nu = 1, 2, \dots, \lambda - 2$ , or  $\lambda - 1$ . Already for the  $\lambda = 3$  case, which we did fully analyze, one gets a lot of different types of spectra. Here, we will merely sketch their classification and provide some examples, leaving the full discussion for a forthcoming publication [25].

Starting with the nondegenerate case, it can easily be shown that the ground states in  $\mathcal{F}_0$ ,  $\mathcal{F}_1$ , and  $\mathcal{F}_2$  may be ordered in three different ways, which we will refer to as I, II, and III, respectively, as listed hereafter

$$\begin{aligned} \text{(I)} \quad & E_0 < E_1 < E_2 \quad \text{if } -1 < \alpha_0 < 2 \text{ and } -2 - \alpha_0 < \alpha_1, \\ \text{(II)} \quad & E_0 < E_2 < E_1 \quad \text{if } 2 < \alpha_0 \text{ and } -4 < \alpha_1, \\ \text{(III)} \quad & E_2 < E_0 < E_1 \quad \text{if } 2 < \alpha_0 \text{ and } -2 - \alpha_0 < \alpha_1 < -4. \end{aligned} \tag{18}$$

Comparing now the positions of the excited states leads one to divide each class into subclasses, themselves labelled by one or two integer indices: (I.1. $n$ ), (I.2. $n$ ), (II.1. $m.n$ ), (II.2. $m.n$ ), (III.1. $m.n$ ), (III.2. $m.n$ ), where  $m, n = 1, 2, 3, \dots$ . Type (I.1. $n$ ) spectra, for instance, correspond to the following ordering and parameter values

$$\begin{aligned} \text{(I.1.}n\text{)} \quad & E_0 < E_3 < \dots < E_{3n-3} < E_1 < E_2 < E_{3n} < E_4 < E_5 < \dots \\ & \text{if } -1 < \alpha_0 < 2 \text{ and } 6n - \alpha_0 - 8 < \alpha_1 < 6n - 4. \end{aligned} \tag{19}$$

We note that only for  $n = 1$ , such an ordering coincides with that of the standard harmonic oscillator.

Considering next the doubly-degenerate case, one again finds various possibilities by reviewing all the limiting cases of the nondegenerate one: (I. $n$ .a), (I. $n$ .b), (II. $m.n$ .a),

(II.*m.n.b*), (II.*m.n.c*), (III.*m.n.a*), (III.*m.n.b*), (III.*m.n.c*), where  $m, n = 1, 2, 3, \dots$ , and a, b, c refer to  $\mathcal{F}_0\text{--}\mathcal{F}_1$ ,  $\mathcal{F}_0\text{--}\mathcal{F}_2$ , and  $\mathcal{F}_1\text{--}\mathcal{F}_2$  degeneracies, respectively. Type (I.*n.a*) spectra, for instance, are given by

$$\begin{aligned} \text{(I.}n\text{.a)} \quad & E_0 < E_3 < \dots < E_{3n-3} < E_{3n} = E_1 < E_2 < E_{3n+3} = E_4 < E_5 < \dots \\ & \text{if } -1 < \alpha_0 < 2 \text{ and } \alpha_1 = 6n - \alpha_0 - 2. \end{aligned} \quad (20)$$

The triply-degenerate case is finally dealt with in a similar way by starting from the doubly-degenerate one. Here, one only gets three possibilities, referred to as (I.*n.abc*), (II.*m.n.abc*), (III.*m.n.abc*), where  $m, n = 1, 2, 3, \dots$ . For instance,

$$\begin{aligned} \text{(I.}n\text{.abc)} \quad & E_0 < E_3 < \dots < E_{3n-3} < E_{3n} = E_1 = E_2 < E_{3n+3} = E_4 = E_5 < \dots \\ & \text{if } \alpha_0 = 2 \text{ and } \alpha_1 = 6n - 4. \end{aligned} \quad (21)$$

We will now show that some of these spectra occur in SSQM when considering cyclic shape invariant potentials [7]. The Hamiltonians corresponding to such potentials have an infinite number of periodically spaced eigenvalues or, in other words, the level spacings are given by  $\omega_0, \omega_1, \dots, \omega_{\lambda-1}, \omega_0, \omega_1, \dots, \omega_{\lambda-1}, \omega_0, \omega_1, \dots$ . The ground state energy vanishing, the general formula for the excited energy levels is  $k\Omega_\lambda + \sum_{\nu=0}^{\mu} \omega_\nu$ , where  $k = 0, 1, 2, \dots$ ,  $\mu = 0, 1, \dots, \lambda - 1$ , and  $\Omega_\lambda \equiv \sum_{\mu=0}^{\lambda-1} \omega_\mu$ .

From  $\lambda = 3$  onwards, the shifted and rescaled Hamiltonian

$$H'_0 \equiv \frac{\Omega_\lambda}{\lambda} (H_0 - E_{gs}), \quad (22)$$

where  $H_0$  is given by Eq. (14), and  $E_{gs}$  denotes its ground state energy, has the same type of spectrum for some parameter values. For  $\lambda = 3$  for instance, one finds that the generic case  $\omega_0 \neq \omega_1 \neq \omega_2 \neq \omega_0$  is obtained for the (I.1.1), (II.1.1.1), and (III.1.1.1) type spectra, corresponding to the orderings  $E_{gs} = E_0 < E_1 < E_2 < E_3 < E_4 < E_5 < \dots$ ,  $E_{gs} = E_0 < E_2 < E_1 < E_3 < E_5 < E_4 < \dots$ ,  $E_{gs} = E_2 < E_0 < E_1 < E_5 < E_3 < E_4 < \dots$ , and the parameter values  $(-1 < \alpha_0 < 2, -2 - \alpha_0 < \alpha_1 < 2)$ ,  $(2 < \alpha_0 < 8, -4 < \alpha_1 < 4 - \alpha_0)$ ,  $(2 < \alpha_0 < 8, -2 - \alpha_0 < \alpha_1 < -4)$ , respectively. An example for each type is displayed on Fig. 1.

It is not surprising that for  $\lambda = 2$ , the Hamiltonian  $H'_0$  does not follow the general rule valid for  $\lambda \geq 3$ . It is indeed well known to be equivalent to the two-particle Calogero Hamiltonian [19, 20], whereas Gangopadhyaya and Sukhatme [8] established that  $\lambda = 2$  cyclic shape invariant potentials include in addition a  $\delta$ -function singularity at  $x = 0$ .



## 4 Bosonization of parasupersymmetric quantum mechanics

From the results of the previous section, it is clear that the bosonic oscillator Hamiltonian  $H_0$ , associated to the algebra  $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$ , has  $\lambda$  series of levels, which if properly shifted with respect to one another, can be made to coincide at least starting from some excited state. Such a spectrum being reminiscent of that of PSSQM Hamiltonians of order  $p = \lambda - 1$ , this hints at a possibility of describing PSSQM in terms of solely boson-like particles, instead of a combination of bosons and parafermions of order  $p$ , as is usually the case [9, 10]. In support to this idea, one should keep in mind that the Calogero-Vasiliev algebra  $\mathcal{A}_{\alpha_0}^{(2)}$  provides a bosonization of ordinary SSQM [17, 20], which is nothing else than PSSQM of order one. Here, we will restrict ourselves to the case of PSSQM of order two, corresponding to the algebra  $\mathcal{A}_{\alpha_0\alpha_1}^{(3)}$ , but our results can be generalized to arbitrary order  $p$  [26].

Let us recall that in PSSQM of order two, the parasupercharge operators  $Q$ ,  $Q^\dagger$ , and the parasupersymmetric Hamiltonian  $\mathcal{H}$  obey the relations

$$Q^3 = 0 \quad (\text{with } Q^2 \neq 0), \quad [\mathcal{H}, Q] = 0, \quad (23)$$

and either

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4Q\mathcal{H}, \quad (24)$$

or

$$[Q, [Q^\dagger, Q]] = 2Q\mathcal{H}, \quad (25)$$

according to whether one chooses Rubakov-Spiridonov [9], or Beckers-Debergh [10] approach. They also satisfy the Hermitian conjugated relations, which we shall take as understood in the remainder of this section.

As ansätze for the operators  $Q$  and  $\mathcal{H}$ , let us choose

$$Q = \sum_{\nu=0}^{\lambda-1} (\xi_\nu a + \eta_\nu a^\dagger) P_\nu, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{\lambda-1} r_\nu P_\nu, \quad (26)$$

where  $\xi_\nu$ ,  $\eta_\nu$  are some complex constants, and  $r_\nu$  some real ones, to be selected in such a way that Eqs. (23), and (24) [or (25)] are satisfied.

Inserting the expression of  $Q$ , given in Eq. (26), into the first condition in Eq. (23), one obtains some restrictions on the parameters  $\xi_\nu$ ,  $\eta_\nu$ , leading to two sets of three independent solutions for  $Q$ . The solutions belonging to the first set may be distinguished by an index  $\mu \in \{0, 1, 2\}$ , and are given by

$$Q_\mu = \left( \xi_{\mu+1} a + \eta_{\mu+1} a^\dagger \right) P_{\mu+1} + \eta_{\mu+2} a^\dagger P_{\mu+2}, \quad \eta_{\mu+1}, \eta_{\mu+2} \neq 0, \quad (27)$$

while those belonging to the second set can be obtained from the former by interchanging the roles of  $Q$  and  $Q^\dagger$ , and will therefore be omitted.

Let us consider next the second condition in Eq. (23) with  $Q_\mu$  given by Eq. (27) for some  $\mu \in \{0, 1, 2\}$ , and with the corresponding parasupersymmetric Hamiltonian  $\mathcal{H}_\mu$  also indexed by  $\mu$ . After some straightforward algebra, one gets the restrictions

$$\xi_{\mu+1} = 0, \quad r_\mu = -2 + \alpha_{\mu+1} + r_{\mu+2}, \quad r_{\mu+1} = 2 - \alpha_\mu + r_{\mu+2}, \quad (28)$$

so that at this stage one is left with three arbitrary constants  $\eta_{\mu+1}$ ,  $\eta_{\mu+2}$ ,  $r_{\mu+2}$ , the first two being complex, and the third real.

It now remains to impose the third condition (24) or (25). To simultaneously deal with both possibilities, let us consider the more general condition

$$u_\mu Q_\mu^2 Q_\mu^\dagger + v_\mu Q_\mu Q_\mu^\dagger Q_\mu + w_\mu Q_\mu^\dagger Q_\mu^2 = 4Q_\mu \mathcal{H}_\mu, \quad (29)$$

where  $u_\mu$ ,  $v_\mu$ ,  $w_\mu$  are some complex constants. Equations (24) and (25) correspond to  $u_\mu = v_\mu = w_\mu = 1$ , and  $u_\mu = w_\mu = -\frac{1}{2}v_\mu = -2$ , respectively.

It can be shown that when taking the previous restrictions (27), (28) into account, Equation (29) can be satisfied for two different choices of  $\{u_\mu, v_\mu, w_\mu\}$  provided some new additional conditions are fulfilled:

$$u_\mu = v_\mu = w_\mu = \frac{4}{|\eta_{\mu+2}|^2 + |\eta_{\mu+1}|^2} \quad \text{if} \quad r_{\mu+2} = (1 + \alpha_{\mu+2}) \frac{|\eta_{\mu+2}|^2 - |\eta_{\mu+1}|^2}{|\eta_{\mu+2}|^2 + |\eta_{\mu+1}|^2}, \quad (30)$$

or

$$\begin{aligned} u_\mu &\neq \frac{4}{|\eta_{\mu+2}|^2 + |\eta_{\mu+1}|^2}, & v_\mu &= \frac{1}{|\eta_{\mu+2}|^2} (4 - |\eta_{\mu+1}|^2 u_\mu), \\ w_\mu &= \frac{1}{|\eta_{\mu+2}|^4} [4 (|\eta_{\mu+2}|^2 - |\eta_{\mu+1}|^2) + |\eta_{\mu+1}|^4 u_\mu] & \text{if} & \quad 1 + \alpha_{\mu+2} = r_{\mu+2} = 0. \end{aligned} \quad (31)$$

The first solution, given in Eq. (30), exists for any algebra  $\mathcal{A}_{\alpha_0\alpha_1}^{(3)}$ , since no condition, other than Eq. (11), is imposed on the parameters  $\alpha_0, \alpha_1$ . Such a solution is of Rubakov-Spiridonov type, as the resulting relation (29) only differs from Eq. (24) by a renormalization of the parasupercharge operators. Let us choose  $|\eta_{\mu+2}| = (4 - |\eta_{\mu+1}|^2)^{1/2}$  so that Eqs. (24) and (29) coincide, and let us fix the overall arbitrary phase of  $Q_\mu$  in such a way that  $\eta_{\mu+1}$  is real and positive (hence it belongs to the interval  $(0, 2)$ ). We then conclude that the two-parameter family of operators

$$\begin{aligned} Q_\mu(\eta_{\mu+1}, \varphi) &= a^\dagger \left( \eta_{\mu+1} P_{\mu+1} + e^{i\varphi} \sqrt{4 - \eta_{\mu+1}^2} P_{\mu+2} \right), \\ \mathcal{H}_\mu(\eta_{\mu+1}) &= N + \frac{1}{2} (2\gamma_{\mu+2} + r_{\mu+2} - 1) I + 2P_{\mu+1} + P_{\mu+2}, \end{aligned} \quad (32)$$

where  $r_{\mu+2} = (1 + \alpha_{\mu+2}) \left( 1 - \frac{1}{2} \eta_{\mu+1}^2 \right)$ ,  $0 < \eta_{\mu+2} < 2$ , and  $0 \leq \varphi < 2\pi$ , can be associated to the Rubakov-Spiridonov PSSQM of order two.

For a given PSSQM Hamiltonian, i.e., for a given  $\eta_{\mu+1}$  value, for instance  $\eta_{\mu+1} = \sqrt{2}$ , we obtain  $r_{\mu+2} = 0$ , and

$$\begin{aligned} Q_\mu(\varphi) &= a^\dagger \sqrt{2} (P_{\mu+1} + e^{i\varphi} P_{\mu+2}), \\ \mathcal{H}_\mu &= N + \frac{1}{2} (2\gamma_{\mu+2} - 1) I + 2P_{\mu+1} + P_{\mu+2}. \end{aligned} \quad (33)$$

From the supercharge operators given in Eq. (33), we may single out the two linear combinations with real coefficients

$$Q_{1\mu} \equiv Q_\mu(0) = a^\dagger \sqrt{2} (P_{\mu+1} + P_{\mu+2}), \quad Q_{2\mu} \equiv Q_\mu(\pi) = a^\dagger \sqrt{2} (P_{\mu+1} - P_{\mu+2}), \quad (34)$$

which may be taken as the two independent conserved parasupercharges with  $p = 2$ , whose existence was established by Khare [27]. It can indeed be checked that  $Q_{1\mu}$  and  $Q_{2\mu}$  not only satisfy Eqs. (23) and (24), but also the mixed trilinear equations given in Ref. [27], and involving two appropriately chosen bosonic constants.

The spectra of the PSSQM Hamiltonians  $\mathcal{H}_\mu$ ,  $\mu = 0, 1, 2$ , defined in Eq. (33), and the action of the corresponding parasupercharge operators  $Q_\mu^\dagger$  are schematically illustrated on Fig. 2. For convenience sake, the ground states of the three spectra have been drawn on the same level, although their energies are in general different, since they are given by

$(2\gamma_2 - 1)/2$ ,  $(2\gamma_0 + 1)/2$ , and  $(2\gamma_1 + 3)/2$  for  $\mu = 0, 1$ , and  $2$ , respectively. For  $\mu = 0$ , parasupersymmetry is unbroken, and the ground state energy may be positive, null, or negative as  $\gamma_2 > -1$  (as a consequence of Eqs. (11), and (16)), whereas for  $\mu = 1$  or  $2$ , parasupersymmetry is broken, and the ground state energy is positive since  $\gamma_0 > -1/2$ , and  $\gamma_1 > -3/2$ . We therefore recognize three of the five possible forms of Rubakov-Spiridonov PSSQM spectra, as given in Figs. 1 (a), (b), (e) of Ref. [9].

Contrary to the first solution of Eq. (29), the second one, given in Eq. (31), only exists for some algebras  $\mathcal{A}_{\alpha_0\alpha_1}^{(3)}$ , namely those for which  $\alpha_{\mu+2} = -1$  (hence the value  $\mu = 1$  is excluded). Since the conditions of Eq. (31) imply that  $v_\mu \neq u_\mu$ , one may consider the possibility of realizing Beckers-Debergh PSSQM. For  $|\eta_{\mu+2}| = |\eta_{\mu+1}|$ , and  $u_\mu = -4/|\eta_{\mu+1}|^2$ , one indeed obtains Eq. (25) up to a renormalization of the parasupercharge operators. If we choose  $|\eta_{\mu+2}| = |\eta_{\mu+1}| = \sqrt{2}$  so that Eqs. (25) and (29) coincide, and if we fix the overall arbitrary phase of  $Q_\mu$  in the same way as in the previous case, we recover Eq. (33), already obtained from Eq. (30). Hence, Equation (31) does not lead to a new realization of PSSQM, but merely shows that for those algebras for which  $\alpha_{\mu+2} = -1$ , Equations (24) and (25) are simultaneously valid, i.e.,  $QQ^\dagger Q = Q^2 Q^\dagger + Q^\dagger Q^2 = 2Q\mathcal{H}$  [10].

## 5 Conclusion

In the present letter, we introduced  $C_\lambda$ -extended oscillator algebras  $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$ ,  $\lambda = 2, 3, \dots$ , containing the Calogero-Vasiliev algebra as a special case (corresponding to  $\lambda = 2$ ). We studied their realization as GDOAs, and their corresponding Fock-space representation.

We then considered the bosonic oscillator Hamiltonian  $H_0$  associated to  $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$ , and proved that for  $\lambda = 3$ , its spectrum has a very rich structure, contrary to what happens for  $\lambda = 2$ . In particular, we showed that for some parameter values, one gets periodic spectra similar to those arising in SSQM with cyclic shape invariant potentials of period three. Finally, we established that  $\mathcal{A}_{\alpha_0\alpha_1}^{(3)}$  provides a bosonization of Rubakov-Spiridonov PSSQM of order two. As mentioned in previous sections, such results may be generalized to higher  $\lambda$  values.

It is clear that there remain many open problems for future study. One of them would be a better understanding of the relationship between  $C_\lambda$ -extended oscillator algebras and SSQM with cyclic shape invariant potentials. Another would be an SSQM interpretation of the  $H_0$  spectrum for some of those parameter values that do not correspond to cyclic shape invariant potentials.

Possible connections with other extensions of SSQM than PSSQM are also worth investigating. Realizations of pseudosupersymmetric [28] and orthosupersymmetric [29] quantum mechanics are under current study, and we hope to report on them in a near future.

## References

- [1] J. M. Leinaas and J. Myrheim, *Nuovo Cimento B* 37 (1977) 1.
- [2] F. Wilczek, *Fractional statistics and anyon superconductivity* (World Scientific, Singapore, 1990);  
R. Iengo and K. Lechner, *Phys. Rep.* 213 (1992) 179.
- [3] H. S. Green, *Phys. Rev.* 90 (1953) 270;  
Y. Ohnuki and S. Kamefuchi, *Quantum field theory and parastatistics* (Springer, Berlin, 1982).
- [4] E. Witten, *Nucl. Phys. B* 185 (1981) 513.
- [5] L. E. Gendenshtein, *JETP Lett.* 38 (1983) 356.
- [6] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* 251 (1995) 267.
- [7] U. P. Sukhatme, C. Rasinariu and A. Khare, *Phys. Lett. A* 234 (1997) 401.
- [8] A. Gangopadhyaya and U. P. Sukhatme, *Phys. Lett. A* 224 (1996) 5.
- [9] V. A. Rubakov and V. P. Spiridonov, *Mod. Phys. Lett. A* 3 (1988) 1337.
- [10] J. Beckers and N. Debergh, *Nucl. Phys. B* 340 (1990) 767.
- [11] V. G. Drinfeld, in: *Proc. Int. Congress of Mathematicians, Berkeley, CA, 1986*, ed. A. M. Gleason (AMS, Providence, RI, 1987) p. 798;  
M. Jimbo, *Lett. Math. Phys.* 10 (1985) 63; 11 (1986) 247.
- [12] J. Katriel and C. Quesne, *J. Math. Phys.* 37 (1996) 1650.
- [13] O. W. Greenberg, *Phys. Rev. Lett.* 64 (1990) 705; *Phys. Rev. D* 43 (1991) 4111;  
D. I. Fivel, *Phys. Rev. Lett.* 65 (1990) 3361.
- [14] S. Chaturvedi and V. Srinivasan, *Phys. Rev. A* 44 (1991) 8024;  
A. J. Macfarlane, *J. Math. Phys.* 35 (1994) 1054.

- [15] C. Quesne, J. Phys. A 27 (1994) 5919; Phys. Lett. A 193 (1994) 245.
- [16] S. Meljanac and M. Mileković, Int. J. Mod. Phys. A 11 (1996) 1391.
- [17] M. S. Plyushchay, Ann. Phys. (NY) 245 (1996) 339.
- [18] M. A. Vasiliev, Int. J. Mod. Phys. A 6 (1991) 1115.
- [19] A. P. Polychronakos, Phys. Rev. Lett. 69 (1992) 703;  
L. Brink, T. H. Hansson and M. A. Vasiliev, Phys. Lett. B 286 (1992) 109;  
L. Brink and M. A. Vasiliev, Mod. Phys. Lett. A 8 (1993) 3585.
- [20] T. Brzeziński, I. L. Egusquiza and A. J. Macfarlane, Phys. Lett. B 311 (1993) 202.
- [21] D. Bonatsos and C. Daskaloyannis, Phys. Lett. B 307 (1993) 100.
- [22] J. F. Cornwell, Vol. 1. Group theory in physics (Academic, New York, 1984).
- [23] C. Quesne and N. Vansteenkiste, J. Phys. A 28 (1995) 7019.
- [24] C. Quesne and N. Vansteenkiste, Helv. Phys. Acta 69 (1996) 141; Czech. J. Phys. 47 (1997) 115.
- [25] C. Quesne and N. Vansteenkiste,  $C_3$ -extended harmonic oscillator Hamiltonian and period-three spectra, Université Libre de Bruxelles preprint (1998).
- [26] C. Quesne and N. Vansteenkiste,  $C_\lambda$ -extended oscillators algebras and some of their deformations, in preparation.
- [27] A. Khare, J. Phys. A 25 (1992) L749; J. Math. Phys. 34 (1993) 1277.
- [28] J. Beckers, N. Debergh and A. G. Nikitin, Fortschr. Phys. 43 (1995) 67, 81;  
J. Beckers and N. Debergh, Int. J. Mod. Phys. A 10 (1995) 2783.
- [29] A. K. Mishra and G. Rajasekaran, Pramana (J. Phys.) 36 (1991) 537;  
A. Khare, A. K. Mishra and G. Rajasekaran, Int. J. Mod. Phys. A 8 (1993) 1245.

## Figure captions

**Fig. 1.** Energy spectra of cyclic shape invariant potentials of period three, obtained with Hamiltonian  $H'_0$ , defined in Eq. (22): (a) type (I.1.1) spectrum with  $\alpha_0 = 0$ ,  $\alpha_1 = 1$ ; (b) type (II.1.1.1) spectrum with  $\alpha_0 = 4$ ,  $\alpha_1 = -3$ ; (c) type (III.1.1.1) spectrum with  $\alpha_0 = 6$ ,  $\alpha_1 = -7$ .

**Fig. 2.** Excited energy spectra of PSSQM Hamiltonians  $\mathcal{H}_\mu$ ,  $\mu = 0, 1, 2$ , defined in Eq. (33). The action of the corresponding parasupercharge operators  $Q_\mu^\dagger$  is also illustrated.



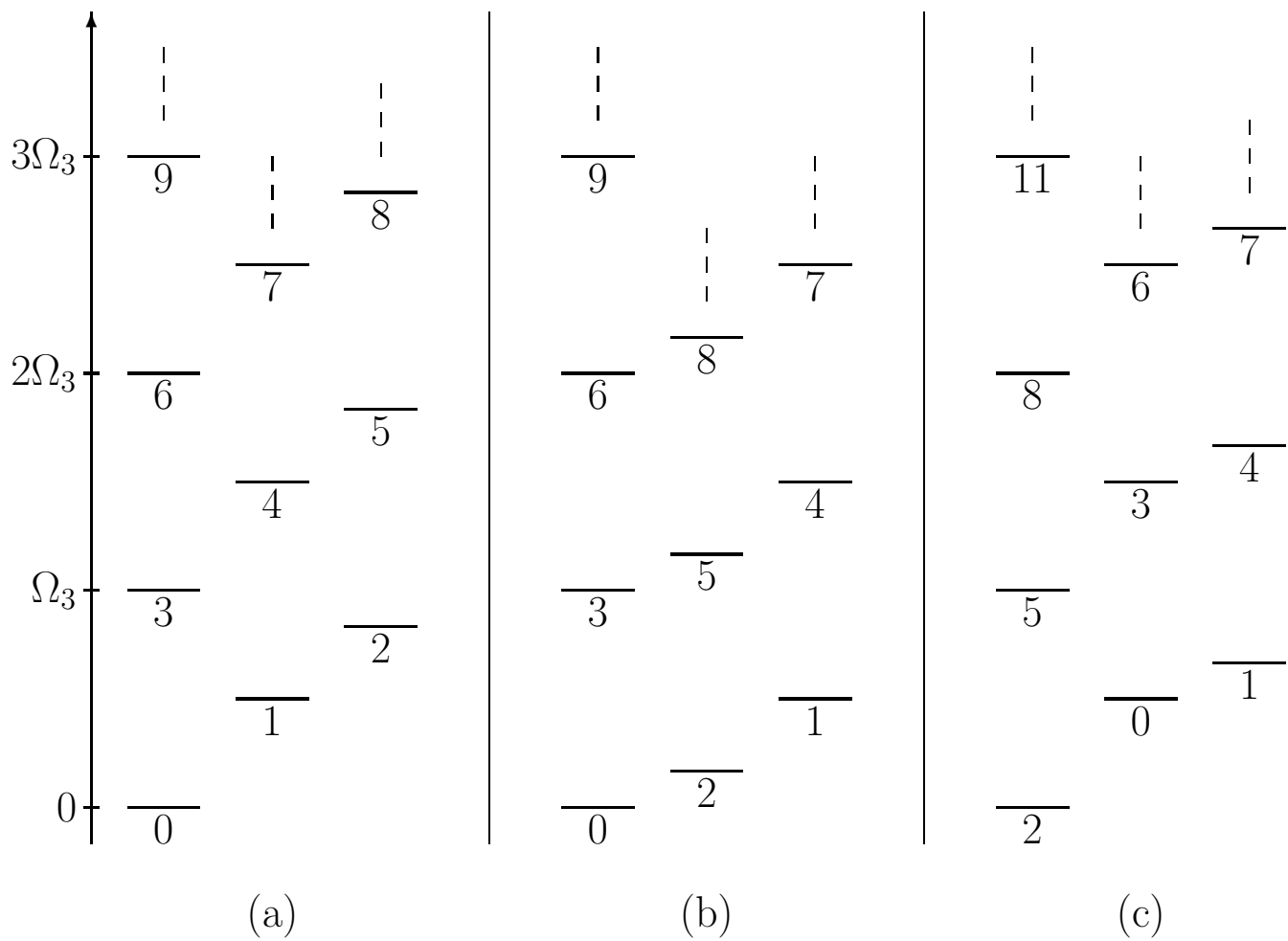


Figure 1

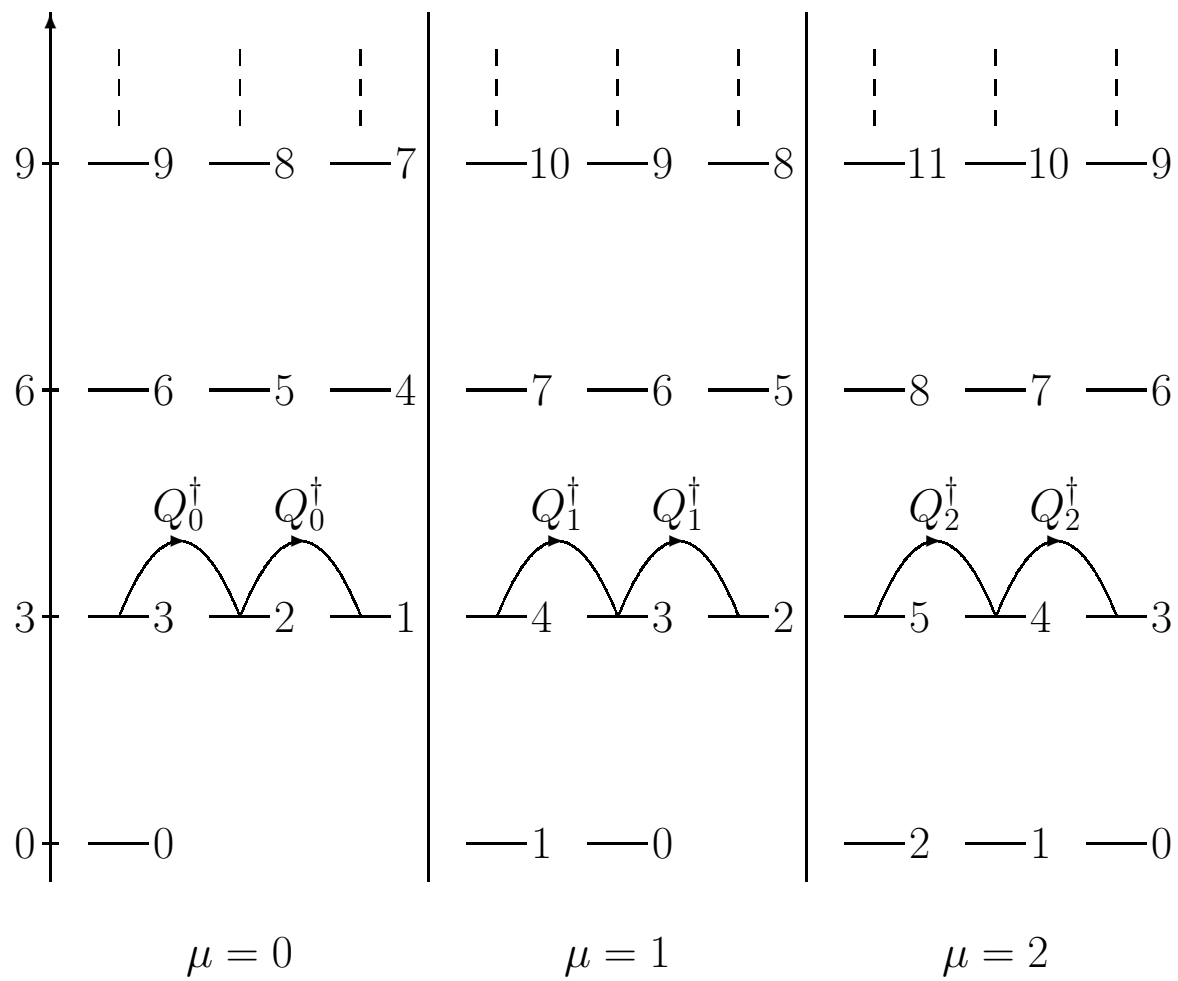


Figure 2